How the world became mathematical

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My title, of course, is an exaggeration. The world no more became mathematical in the seventeenth century than it became ironic in the nineteenth. Either it *was* mathematical all along, and seventeenth-century philosophers discovered it was, or, if it wasn't, it could not have been made so by a few books. What became mathematical was physics, and whether that has any bearing on the furniture of the universe is one topic of this paper. Garber says, and I tend to agree, that for Descartes bodies are the things of geometry made real (✰Ref). That is a claim about the world: what God created, and what we know in physics, is nothing other than *res extensa* and its modes. Others, including Marion after Heidegger, hold that in modern science, here represented at its origins by Descartes, representation displaces beings: the knower no longer confronts Being or beings but rather a system of signs, a "code" as Marion calls it, to which the knower stands in the relation of subject to object. The *Meditations*, or perhaps even the *Regulæ*, are the first step toward the transcendental idealism of Kant.

Most of this paper will be devoted to a more concrete question. Physics in the seventeenth century increasingly became a matter of *applying* mathematical knowledge to the solution of physical problems. The "mixed sciences" of astronomy, optics, and music, sciences then distinct from physics, became models of understanding for all of natural philosophy. My interest here is in one aspect of that development: how particular physical situations are transformed into mathematical problems. That question is distinct from ontological questions about the existence of powers or forces, and from epistemological questions about laws and their confirmation. Typically when considering a particular physical situation or "setup", the philosopher takes for granted—if he considers it at all—the ontology and laws of some enveloping primary physics, neither of which is thought to be directly at stake.

That mathematics *is* applicable to physical questions was of course a philosophical problem in the period, as it is now.[1] But supposing that it is, for each particular *expérience* the *manner* in which mathematical reasoning is to be brought to bear remains a question. Seventeenth-century philosophers like Descartes, Fermat, and Hobbes had, on the one hand, a body of techniques for deriving solutions to mathematical problems; on the other hand, they had descriptions of various *expériences*, some involving artificial setups, others provided gratis by nature. How were the techniques to be made suitable to answer questions about those *expériences*, and how were the *expériences* to be redescribed so as to present the right sorts of questions? The physicist now has a variety of what Kuhn called exemplars, analogy with which helps to solve those problems. The seventeenth-century natural philosopher had very few, if any—those physical situations that later came to be exemplary are in this period just the cases about which decisions had to be made. In looking at Descartes' work in mathematical physics and at his critics, the first requisite is to cease to regard *any* solution as obvious.

1. Falling bodies

The rule of falling bodies is a case in point. Descartes took up the question several times. In his work with Beeckman in the winter of 1618–1619, he derives what we would regard as the correct rule: distance is proportional to the square of the time of descent. The question put to Descartes by Beeckman is: Can one know, given that one knows how far a body has fallen in two hours, how far it fell in the first hour? Starting from the principle, already enunciated by Beeckman some years earlier, that in a vac-

1. Mark Steiner, The applicability of mathematics as a philosophical problem (Harvard, 1998).

uum “what once moves, always moves” (AT 10:60), Descartes shows that the distance traversed by a body moving between time *a* and time *b* (the vertical axis in Fig. 1) is proportional to the area of the triangle *abc*. Since the area of the triangle *afg* is one-third the area of the trapezoid *fbcg*, the answer to Beeckman’s question is that the body will have fallen one-fourth of the total distance in the first hour of its descent.

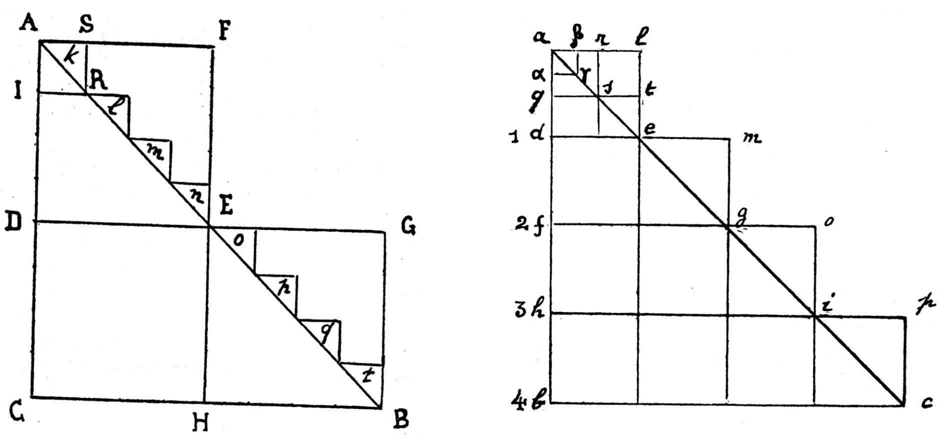

Fig. 1. Diagrams demonstrating the rule of falling bodies. The first is Beeckman’s, the second from a copy of Descartes’ notes (AT 10:59, 76).

Strictly speaking, that is not quite what the diagram shows. Instead it shows us four discrete moments *a, d, f, h* at which new “force” is imparted to the falling body, and, correspondingly, a collection of squares representing the distance travelled in the time from *a* to *d*, *d* to *f*, and so forth. Notice that the upper corners of the right-hand squares in each row stick out of the triangle *abc*. The triangle *abc* is arrived at by imagining the axis of time divided into smaller and smaller moments, until—in the limit, as we would say—the resulting stair-step figure, representing *minima* of time and minimal increments of “force”, does indeed become that triangle. As Beeckman puts it: “if a minimum moment of space were some quantity, then [the velocity of the body] will be an arithmetic progression” (AT 11:61); he seems to be dubious about that supposition, and reassures himself by showing that the ratios between the top and bottom halves of figures like Fig. 1 constructed with ever finer (but not infinitesimal) divisions of the time axis do indeed converge (as we would say) to 1/4. The doubt arises because the physical *cause* of descent is “corporeal spirits”, which in “distinct intervals” pull down the falling body; but because there are so many particles, the convergence argument is *physically* appropriate, and (as Beeckman says) the demonstration using the triangle may stand.[2]

A second version of the rule of falling bodies is found in a letter from Descartes to Mersenne in November 1629. The bulk of the letter is in French, but the demonstration is in Latin, which leads Adam and Tannery to suppose that it dates from the time of Descartes' collaboration with Beeckman. Again we have a diagram which on the face of it resembles the diagrams from that earlier period. Again the demonstration

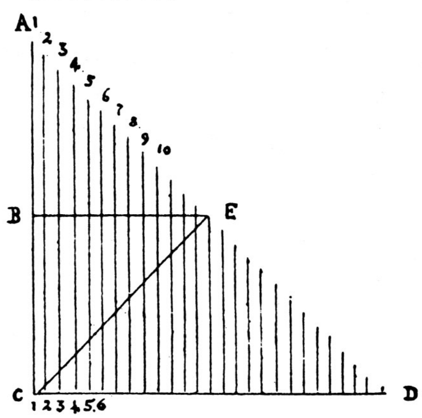

Fig. 2. Diagram demonstrating the rule of falling bodies (AT 1:72).

begins with the principle that "what once begins to move, moves always with equal quickness" (AT 1:72). Every moment "new forces of descent" are added, so that the body traverses the space BC "much more quickly" than the space AB, because it retains "all the impetus by which it moved through the space AB" while continuing to gain new impetus by virtue of its weight.

What follows is not easy to follow. Descartes says that the "first line" denotes the "force of quickness impressed during the first moment", the second the force impressed in the second, and so forth. It is not clear how to square this with the dia-

2. "Sic duorum terminorum progressio, quæ est 1.2., se habet ut 1 ad 3. Sic 1.2.3.4.5.6.7.8 se habet ut 10 ad 36. Sic termini hi octo ad 16 se habent ut 36 ad 136, quod nondum est ut 1 ad 4. Si igitur descensus lapidis fiat per distincta intervalla, trahente terrâ per corporeos spirtus, erunt tamen hæc intervalla sue momenta tam exigue, ut proportio eorum arithmetica, ob multitudinem particularum, non sensibiliter fuerit minor quàm 1 ad 4. Retinenda ergo triangularis dicta demonstratio" (Beeckman Journal 105v; AT 11:61). Descartes, for his part, simply says the temporal minima into which we suppose the time axis to be divided "must be imagined to be indivisible and containing no parts" (11:76).

gram, but Descartes derives, as before, the conclusion that the weight will travel three times further in the second half of the duration of its descent than in the first. Hence "if in three moments it descends from A to B, it descends in one moment from B to C", or, in other words,

> in four moments it makes a journey twice as far as in three, and consequently in twelve moments twice as far as in nine, in sixteen four times as many as nine, and so on (AT 1:73).

Letting *s*(*t*) be the distance travelled in time *t*, we seem to have a functional equation

$$s\left(\frac{4}{3}t\right) = 2s(t)$$

the solution to which (see AT 1:75) is a function *s* of the form

$$s(t) = s(1) \cdot t^{(\log 2)/(\log(4/3))}$$

Needless to say, that is not what we were expecting.

Descartes seems to have thought to calculate the distance ratio $s(t_1)/s(t_2)$ for any ratio t_1/t_2 of times it suffices to calculate it for *one* ratio and then use geometric proportions to calculate it for *any* ratio. That is, at least, what his example suggests. In the diagram, on the other hand, if the vertical axis represents (as Adam and Tannery believe) the *distance* traversed by the body and not the time of descent, then the velocity of the particle would be proportional to the distance travelled. The resulting rule would be exponential in from.[3]

From this we can conclude that Descartes had no preconception concerning the mathematical *form* of the rule. There were no exemplars from which to guess its form by analogy.[4] We can also see that because he thinks in terms of ratios—that being the way of geometers—the *acceleration* of the falling body has no role in his thinking. The rule he is looking for is something like "if the ratio of the first time to the second is such-and-such, then the ratio of the first distance to the second is such-and-such". In a rule of that form, the acceleration drops out. His diagrams, one might now say, are

3. ✫ Reference to analyses that yield an exponential rule. Vuillemin? A third version of the derivation (To Mersenne 14 Aug 1634, AT 1:304) reverts to the reasoning of the first. Descartes by this time had read Galileo's derivation in the Massimi Sistemi of 1632.

4. ✫ Reference: Steiner.

pictures not of a metric but of an affine space in which the unit of measurement can be reduced or enlarged at will.

In short, Descartes brings to the problem posed to him by Beeckman (and again by Mersenne) a collection of mathematical techniques, largely geometrical, that determine a space of possible solution-types noticeably distinct from the space determined by the later techniques of the calculus. That he could propose different derivations shows that, even given the law of inertia (in Beeckman's formulation) *and* the mathematical techniques, the manner in which the phenomenon was to be analyzed remained an open question. Descartes understands that a constant "force" impelling a body yields a constant increment in velocity. But the application even of that correct understanding remains an open question.

2. Optics: the dispute with Fermat

The path along which physics travelled to its mathematical destination was bumpy and only in hindsight unique. Further illustration of this point is provided by the dispute between Descartes and Fermat following the publication of the *Dioptrique* in 1637.[5] In the *Dioptrique* Descartes demonstrates the law of reflection using the following diagram. On the supposition that "the determination to move in one direction or another […] can be divided into all the parts of which one can imagine it to be composed", Descartes divides the "determination to move" AB of a ray of light into the components AC and AH. The component AH parallel to the surface of reflection is unaltered at the point of reflection: it is therefore extended by a segment HF equal to AH. We draw a circle through A centered at the point of reflection, and a line FED perpendicular to the surface CBE of

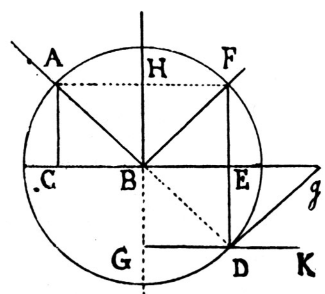

Fig. 3. Diagram for the law of reflection (AT 1:358; cf. AT 6:95).

5. Fermat saw the Dioptrique before its publication, having been sent by Beaugrand a copy of the pages sent to Paris in order to obtain the privilège (AT 1:355).

reflection. The light, Descartes says, must arrive at the circumference of the circle and at the line FED at the same time; it can do so only by travelling to the point F or the point D. Since the surface prevents it from passing toward D, it must instead pass through F (AT 6:97; 1:358).

Fermat rejects the argument. What would prevent us from "imagining that the determination of the ball that moves from A toward B is composed of two others", one perpendicular to the surface of reflection (AC) and the other not parallel to it but directed away from it (AH)? (AT 1:358) If we then follow Descartes' reasoning, we first draw the perpendicular BH from B to H, and then extend AH to F. Evidently the angle of reflection would no longer be equal to the angle of incidence. Fermat concludes that Descartes has chosen from among the infinitely many divisions of the determination of the light just that which suits his purpose. He has "accommodated his *medium* [middle premise] to his conclusion, and we know as little as we did before". Moreover, an "imaginary", arbitrary, division "can never be the cause of a real effect" (AT 1:359).

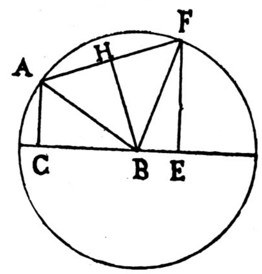

Fig. 4. Diagram for the law of reflection (AT 1:358; cf. AT 6:95).

Descartes is, implicitly and without justification, applying something like the parallelogram rule to the decomposition of velocities or momenta. Fermat is visibly *not* following any such rule. The bizarreness of his alternative decomposition should not, all the same, mislead us as to the force of his objections. They go to the heart of the matter. If geometry is to be applied to the demonstration of the law of reflection, and if that application proceeds by way of decomposition, the arbitrariness of the decomposition, which Descartes admits can be carried out in infinitely many ways, must be eliminated. Moreover, if the component movements are to be admissible in physical (as opposed to merely geometrical) argument, then they must exhibit, so to speak, their credentials: Descartes must show how they can be the cause of *real* effects.

Descartes in his reply answers both criticisms. The division he uses in his proof is not arbitrary, nor are the components imaginary. On the contrary: "there is no reason to conclude that the division of this determination, which is brought about by the surface CBE [see Fig. 3 *supra*], which is a real surface, namely that of the smooth body CBE, is only imaginary" (1:452). The *physical* reason for choosing this rather than some other division is that only movement perpendicular to the surface is entirely opposed by it, and only movement parallel to the surface is entirely unopposed.

In a second set of objections, Fermat considers only Descartes' demonstration of the law of refraction.[6] After conceding, for the sake of argument, that the determination of the light in the direction perpendicular to the surface of refraction is changed, Fermat demonstrates a version of the parallelogram rule, first in the case of perpendicular components and then more generally. The key point is that increasing the angle between the component motions decreases the distance traversed by a body whose movement has been divided into those components. Fermat applies the proposition to show that in Descartes' proof of the law of refraction Descartes is wrongly supposing that an alteration of the angle between the two components of the motion of the light will *not* alter its velocity.

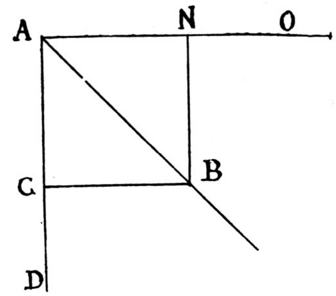

Fig. 5. Composition of movements (AT 1:469).

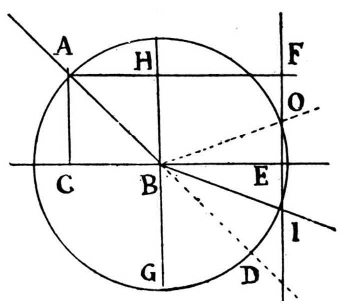

Fig. 6. Law of refraction (AT 1:466).

Descartes' response to what he calls Fermat's "paralogism" is that Pascal has confused the composition of speeds with that of determinations. There are two senses in which a movement can be composed. "In the first sense, only the determination of the movement is composed, and its speed is not, except insofar as it accompanies this

6. "Mais puisque nous ne doutons pas que les reflextions ne se fassent a angles egaux, il est superflu de disputer de la preuve, puisque nous connoissons la verité; & i'estime que ie feray mieux, sans marchander, de venir a la refraction, qui sert de but a la Dioptrique" (Fermat to Mersenne, AT 1:465).

determination” (2:19). In the second sense, which Descartes recognizes as his own, “only the speed of the movement is composed”. A surface of refraction affects only the speed, and so it is only the speed “that follows the laws of composition, and not the determination”, which instead must accommodate itself to the speed (2:20).

It is not my intention to evaluate Fermat’s objection or Descartes’ response. The point to notice here is that the dispute turns, not only on the reality and effectiveness of the components of a divided motion, but on the appropriate point of application for geometrical reasonings that themselves are not in dispute. What, in other words, do the lines (and the corresponding quantities) in diagrams like Fig. 6 represent?

3. Conclusion

Let me return to the questions raised at the beginning of this paper. It seems to me that at a certain level of generality, Garber’s characterization of the world of Cartesian physics is correct. There is nothing more to bodies than their substance, which is *res extensa*, and the modes pertaining to that substance so understood. But when the geometrical knowledge possessed by Descartes and his contemporaries is applied to the solution of physical problems, things are less simple. There are, first, questions of reality. Fermat at first doubts that the “components” of speed or of determination real; Descartes anchors his choice of composition in the reality of the reflecting surface. The geometric description would be the same in either case, but the causal story differs.

There are also questions of coordination. In his first derivation of the rule of falling bodies, Descartes takes time to be what we would call the independent variable; in the second, it is unclear, but the diagram suggests a confusion between time and distance. The speed of a falling body varies both with time and distance travelled, but its dependence on distance is derivative from its dependence on time. Similarly in the debate on refraction, one point at issue is whether the relevant variable in determining the angle of refraction is to be the speed or the determination of the light, or both. Descartes and Fermat seem to be in agreement on both the geometrical facts and the laws of nature. Where they differ is in the manner in which the phenomenon of refraction is to be subsumed under a geometric reasoning.

No doubt what counts as a physical phenomenon is to some degree determined by amenability to mathematical treatment. To that extent not only physics but the world—of physics—becomes mathematical. But, as in the case of falling bodies, the world retains, even for mathematical physics, its prerogative of resisting some of our attempts at representing it.